ALGEBRAIC TOPOLOGY I WS23/24, HOMEWORK SHEET 6

DEADLINE: FRIDAY, NOVEMBER 24TH

Problem 1. Let X be a connected CW-complex with basepoint $x \in X$. Recall that for each $n \geq 1$, $\pi_1(X,x)$ acts on $\pi_n(X,x)$. In the homotopy category, this induces a natural action of $\pi_1 X$ on $K\big(\pi_n(X,x),n\big)$, which further induces an action on homology

$$\alpha : \pi_1(X,x) \times H_* K\big(\pi_n(X,x),n\big) \to H_* K\big(\pi_n(X,x),n\big).$$

Recall that for each $n \geq 2$, the Postnikov tower gives a fibre sequence

$$K(\pi_n X, n) \to \tau_{\leq n} X \to \tau_{\leq n-1} X.$$

Let $f_{n-1} : X \to \tau_{\leq n-1}$ be the canonical map, and let $y = f_{n-1}(x)$. Then $\pi_1(\tau_{\leq n-1}X, y)$ acts on the homology of the homotopy fibre at the point y, i.e., $K(\pi_n(X,x),n)$. Let

$$\beta : \pi_1(\tau_{\leq n-1}X, y) \times H_* K(\pi_n(X,x),n) \to H_* K(\pi_n(X,x),n)$$

denote this action.

Show that the two actions α and β on homology are the same, under the identification

$$(f_{n-1})_* \colon \pi_1(X,x) \xrightarrow{\cong} \pi_1(\tau_{\leq n-1}X, y).$$

Problem 2. As discussed in the lecture, the first p-torsion class in $\pi_* S^3$ is found in degree $2p$. Recall that for all $n \geq 3$, the Hopf map $\eta \colon S^3 \to S^2$ induces an isomorphism $\eta_* \colon \pi_n S^3 \cong \pi_n S^2$. We let $x \in \pi_{2p} S^2$ be a p-torsion class. Consider the suspension $\Sigma x \in \pi_{2p+1} S^3$.

Show that if p is odd, then $\Sigma x = 0$.

Bonus: Show that when $p = 2$, Σx suspends to a generator of $\pi_5 S^3$.